A Stochastic Cellular Automata for Three-Coloring Penrose Tiles

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Abstract

We present a three state, stochastic cellular automata which runs on Penrose tilings and evolves to a three-colored equilibrium.

1 Introduction

In 1973 and 1974, Roger Penrose discovered three sets of polygons each of which tiles the plane aperiodically and (if certain matching conditions are enforced) only aperiodically. Later, John Conway asked if such tilings can be three-colored, where adjacent tiles are to receive different colors. This question has been answered affirmatively for two types of Penrose tilings, but appears to be open for the remaining type. In this paper, we present an algorithm which seems to three-color Penrose tilings of all types. The algorithm works by running a particular three state, stochastic cellular automata on a given Penrose tiling. The cellular automata is chosen so that three-colorings are stable and it seems to generally evolve to such an equilibrium.

2 Penrose tilings

There are three types of Penrose tilings: tilings by kites and darts, tilings by rhombs, and tilings by pentacles. We describe them briefly here. More detailed references are [1] and [2].

2.1 Tilings by kites and darts

Figure 1 shows the kite and dart. The sides have length either 1 or τ, the golden ratio, and the angles are all integer multiples of $\frac{\pi}{5}$. The filled and unfilled disks at the vertices are used to enforce a matching condition. When tiling the plane with kites and darts, we demand that filled disks meet filled disks and unfilled disks meet unfilled disks. This matching condition guarantees that any tiling by kites and darts will be aperiodic, i.e. no translation of the tiling maps each tile to another tile. Figure 2 shows part of such a tiling.

2.2 Tilings by Rhombs

Figure 3 shows the fat and skinny Rhombs. The sides all have length 1 and the angles are all integer multiples of $\frac{\pi}{5}$. The matching condition is slightly more complicated. We demand that filled disks meet filled disks, unfilled disks meet unfilled disks, and oriented edges meet with the correct orientation. Again, this matching condition guarantees that any tiling by rhombs will be aperiodic. Figure 4 shows part of such a tiling.

2.3 Tilings by pentacles

Figure 5 shows the pentacles. Again, all of the sides have length one and the angles are all integer multiples of $\frac{\pi}{5}$. The labels indicate a matching condition, which again assures aperiodicity. The edges labeled 0 must fit against edges labeled $\bar{0}$, 1 against $\bar{1}$, and 2 against $\bar{2}$. Note that the three pentagons are congruent, but have different matching conditions. A portion of a tiling by pentacles is shown in figure 6.

3 Coloring the Tiles

A tiling is called three-colorable if we may assign one of three distinct colors to each tile such that adjacent tiles have different colors. Tiles are said to be adjacent if their intersection is a line segment. Figures 7, 8, and 9 show three-colored tilings by kites and darts, rhombs, and pentacles respectively. Sibley and Wagon [5] prove that tilings by rhombs are three-colorable and William Paulsen has proven that tilings by kites and darts are three-colorable.

Our pictures are the final stage of a three state, stochastic cellular automata which can run on any tiling. The cellular automata works as follows. First, assign one of three possible colors to each tile randomly. Then, allow the cellular automata to evolve according to the following set of rules:

- If the value of a cell (or tile) equals the value of a bordering cell which is closer to the origin (as measured by some arbitrary point chosen within each tile), then with 90% probability, the cell changes value randomly to one of the other two colors.

- If the value of a cell does not equal the value of a bordering cell which is closer to the origin, but does equal the value of a cell farther away from the origin, then with 10% probability, the cell changes value.

- If the value of the cell does not equal the value of any bordering cell, the cell does not change value.

Note that three-colorings are stable under these rules. The hope is that three-colorings are attractive equilibria. Figure 10 demonstrates the algorithm on a small piece of a kite and dart tiling. A larger dynamic view is available at the author's web page:
http://www.unca.edu/˜mcmcclur/mathematicaGraphics/PenroseColoring/

4 Implementation Notes

All the images for this paper were generated with *Mathematica*. The tilings were generated using the **DigraphFractals** *Mathematica* package by the author as described in [3] and [4]. These images were then converted to **PlanarMap** and **PlanarGraph** objects as defined in the **GraphColoring** *Mathematica* package by Stan Wagon [6]. Code to run the cellular automata on the **PlanarGraph** objects was written by the author. Final three-colored

images were rendered by the **ShowMap** function defined in the **GraphColoring** package. All code and more images are available on the authors web page:
http://www.unca.edu/~mcmcclur/mathematicaGraphics/PenroseColoring/

References

[1] Gardner, M., *Penrose Tiles to Trapdoor Ciphers*, W. H. Freeman, New York, 1989.

[2] Grünbaum, B. and Shepard, G. C., *Tilings and Patterns*, W. H. Freeman, New York, 1987.

[3] McClure, M. "Directed-graph iterated function systems." *Mathematica in Education and Research* **9** (2000), no. 2.

[4] McClure, M., Digraph self-similar sets and aperiodic tilings. In preparation.

[5] Sibley, T. and Wagon, S., Rhombic Penrose tilings can be 3-colored. *The American Mathematica Monthly*, 2000, **107** #3, 251-253.

[6] Wagon, S., *Mathematica in Action*, Springer-Verlag, New York, 1999.

5 Legend for the figures

Figure 1 The kite and dart

Figure 2 Part of a tiling by kites and darts

Figure 3 The fat and skinny rhombs

Figure 4 Part of a tiling by rhombs

Figure 5 The pentacles

Figure 6 Part of a tiling by pentacles

Figure 7 Part of a three-colored tiling by kites and darts

Figure 8 Part of a three-colored tiling by rhombs

Figure 9 Part of a three-colored tiling by pentacles

Figure 10 Evolution to a three-coloring of a small piece of a tiling by kites and darts.

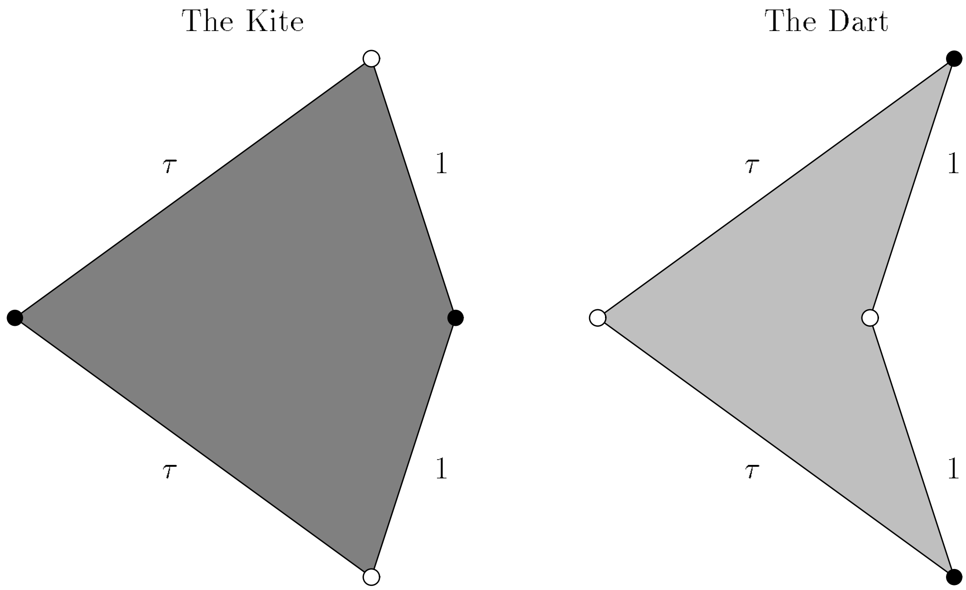

Figure 1: The kite and dart

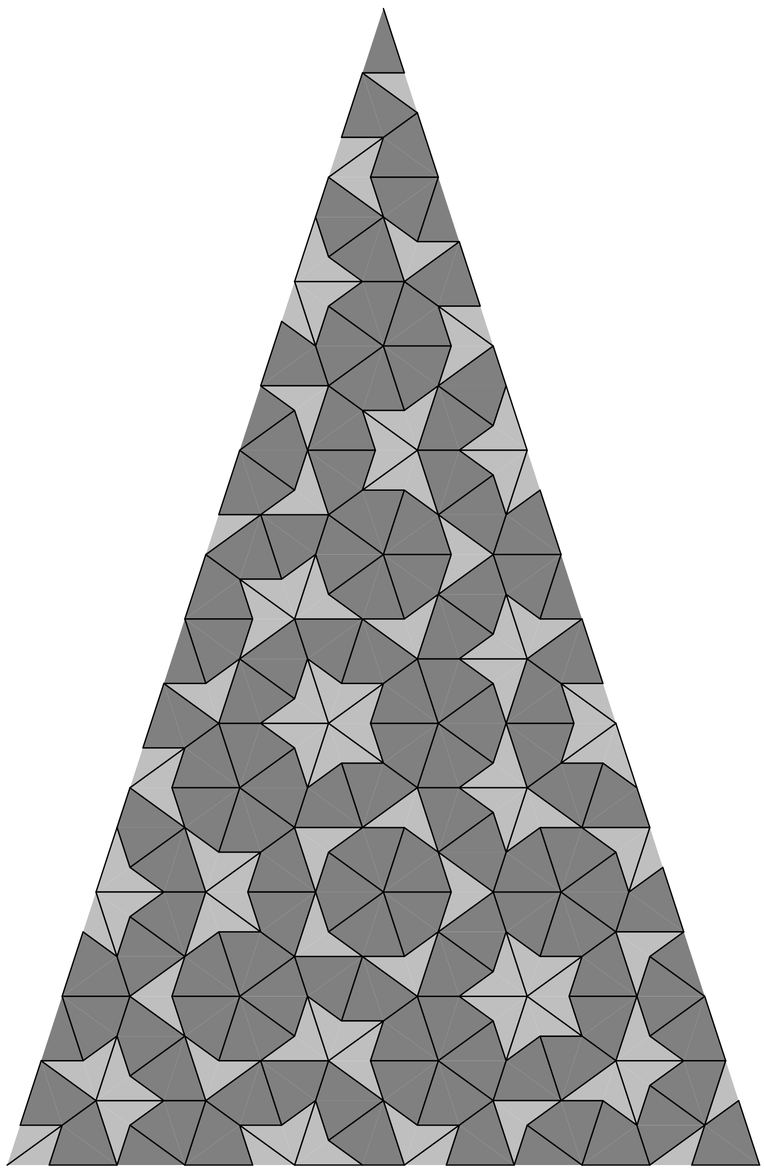

Figure 2: Part of a tiling by kites and darts

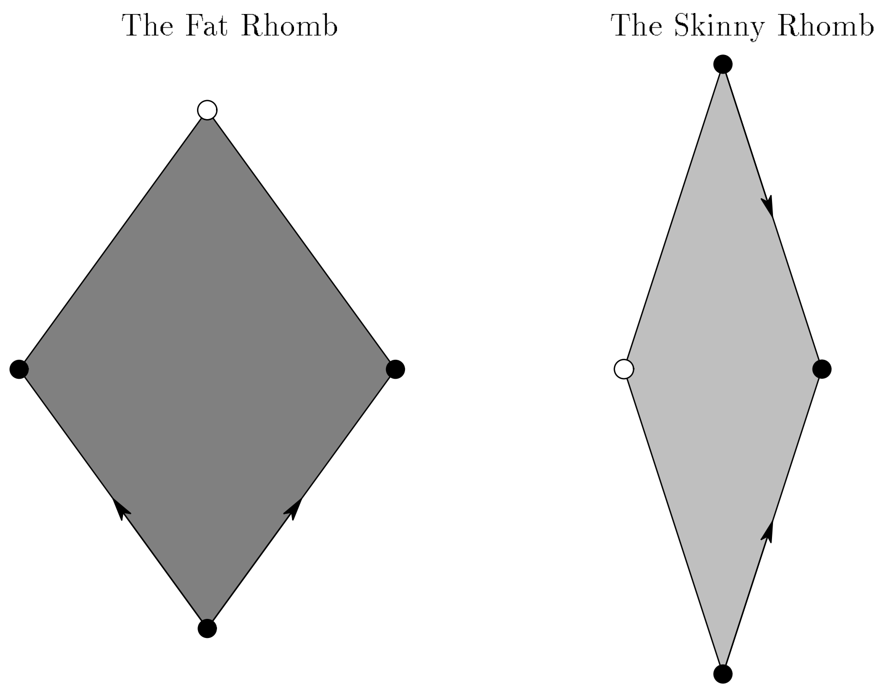

Figure 3: The fat and skinny rhombs

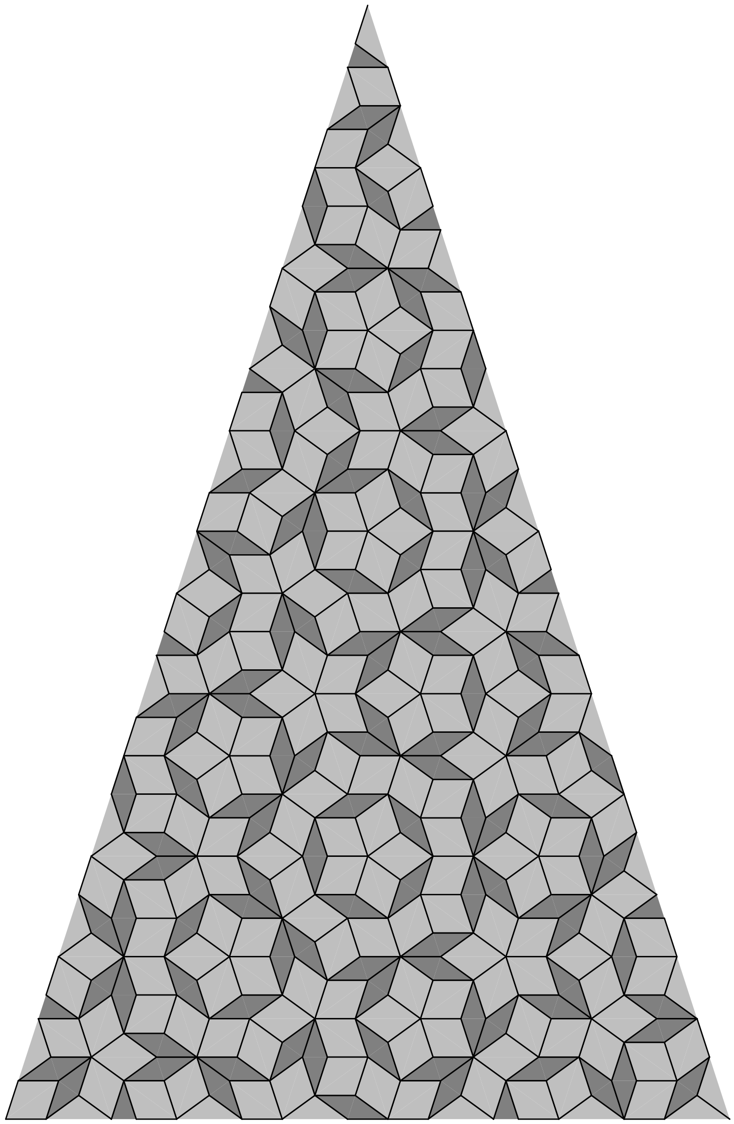

Figure 4: Part of a tiling by rhombs

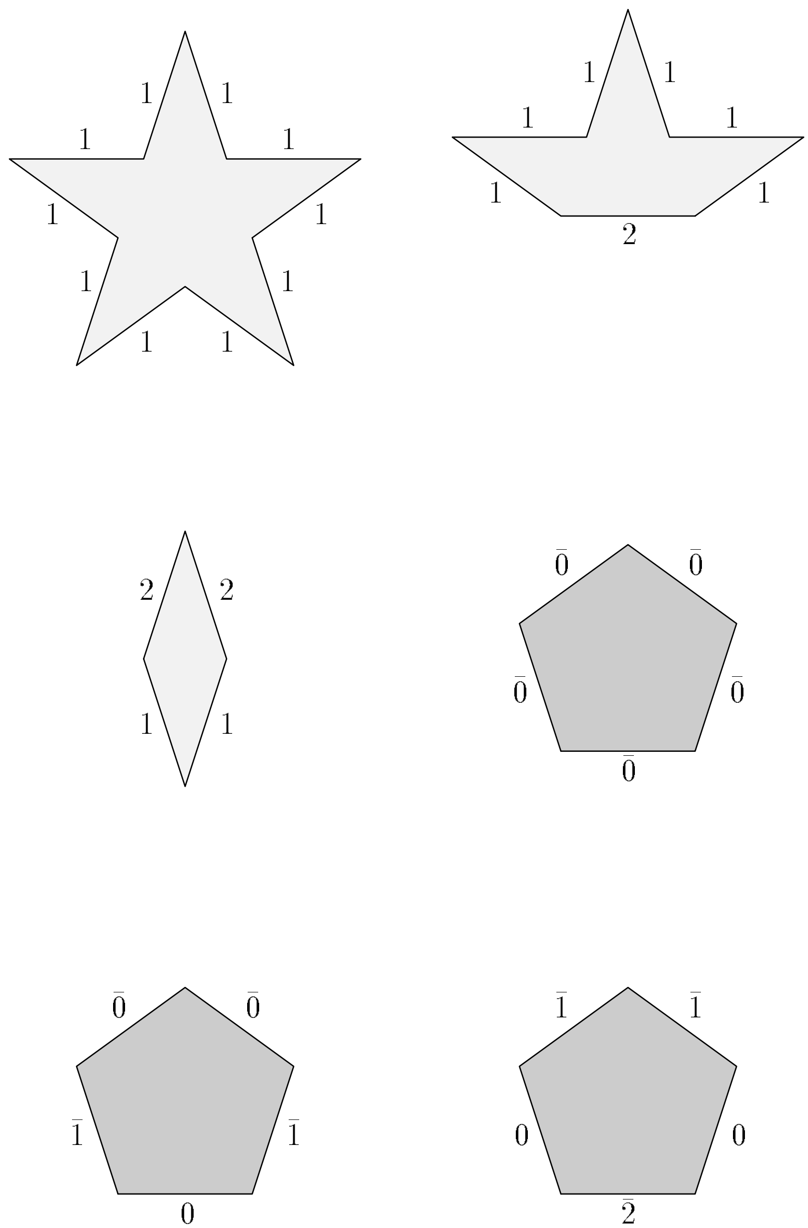

Figure 5: The pentacles

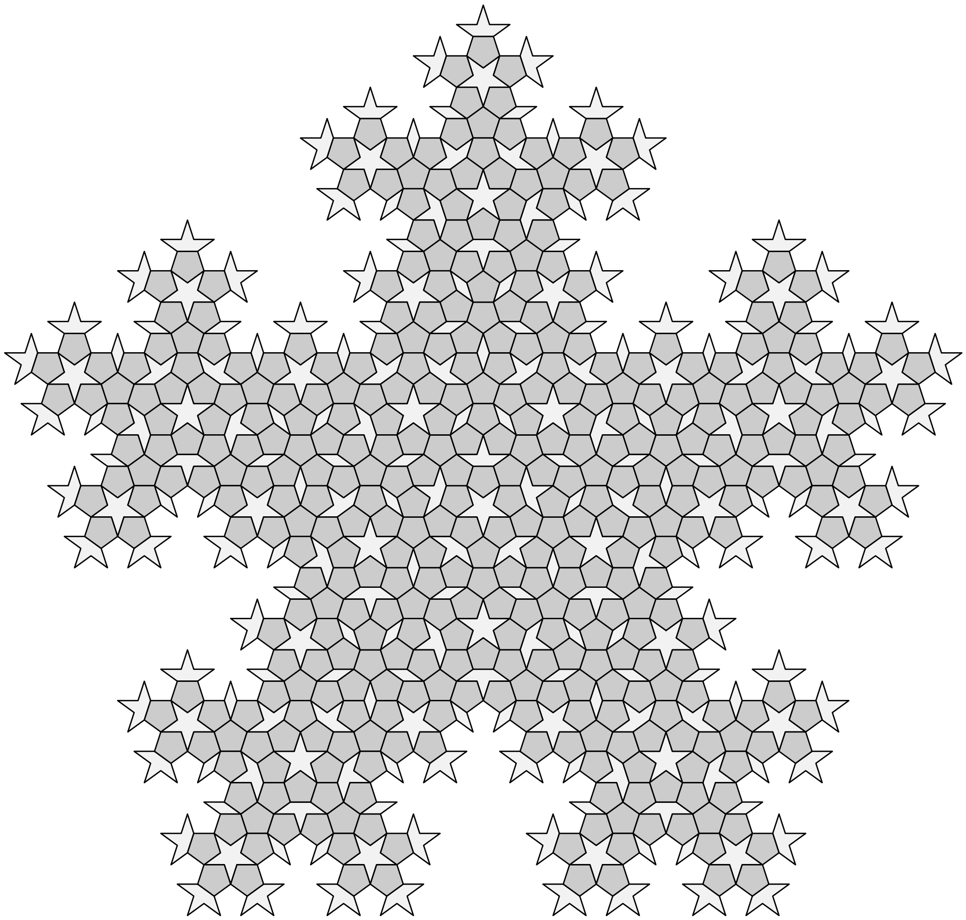

Figure 6: Part of a tiling by pentacles

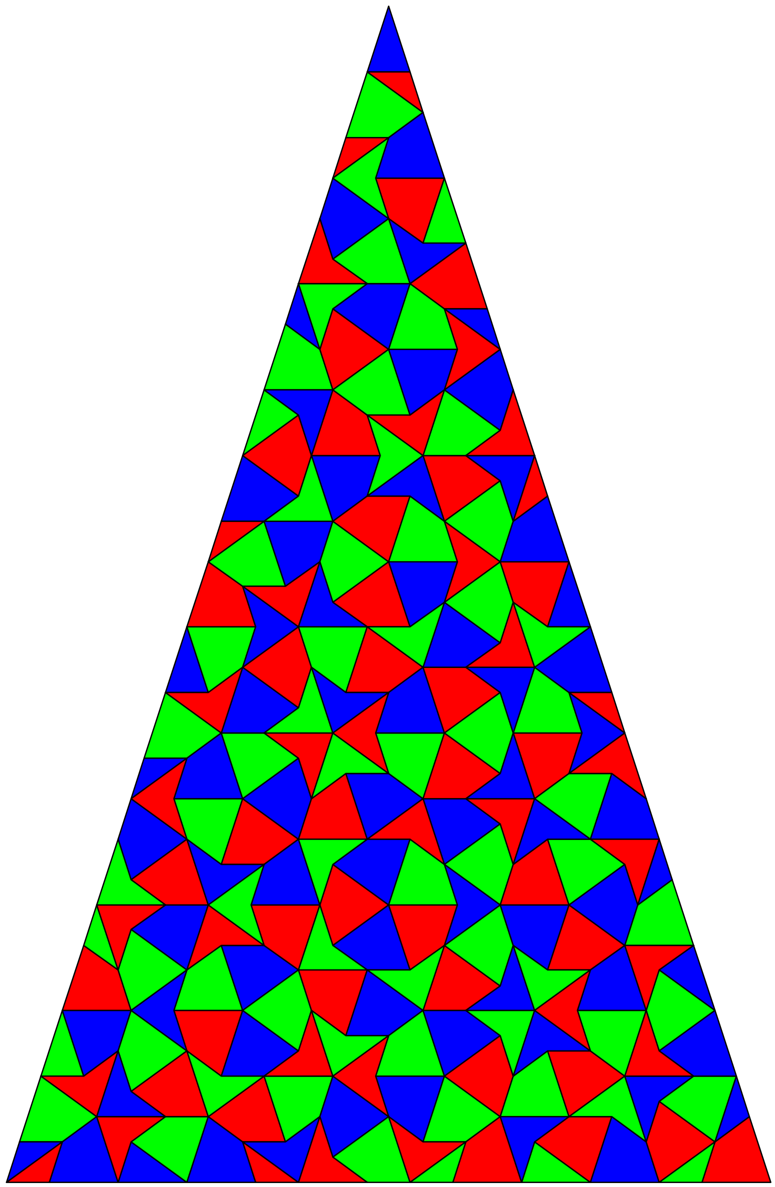

Figure 7: Part of a three-colored tiling by kites and darts

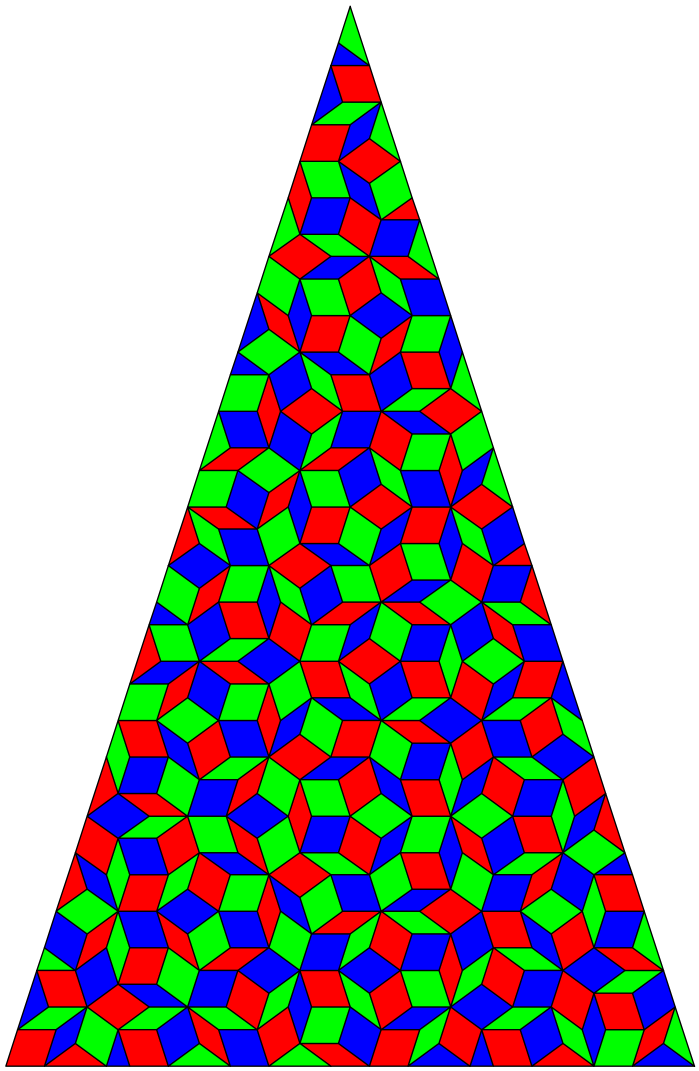

Figure 8: Part of a three-colored tiling by rhombs

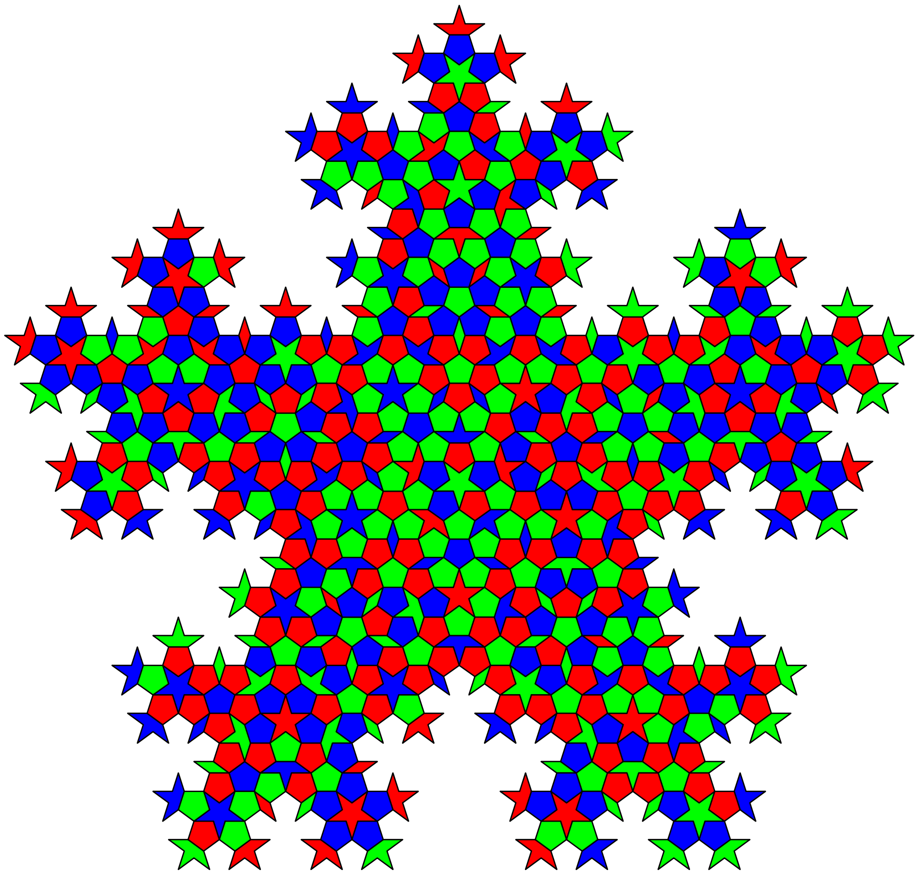

Figure 9: Part of a three-colored tiling by pentacles

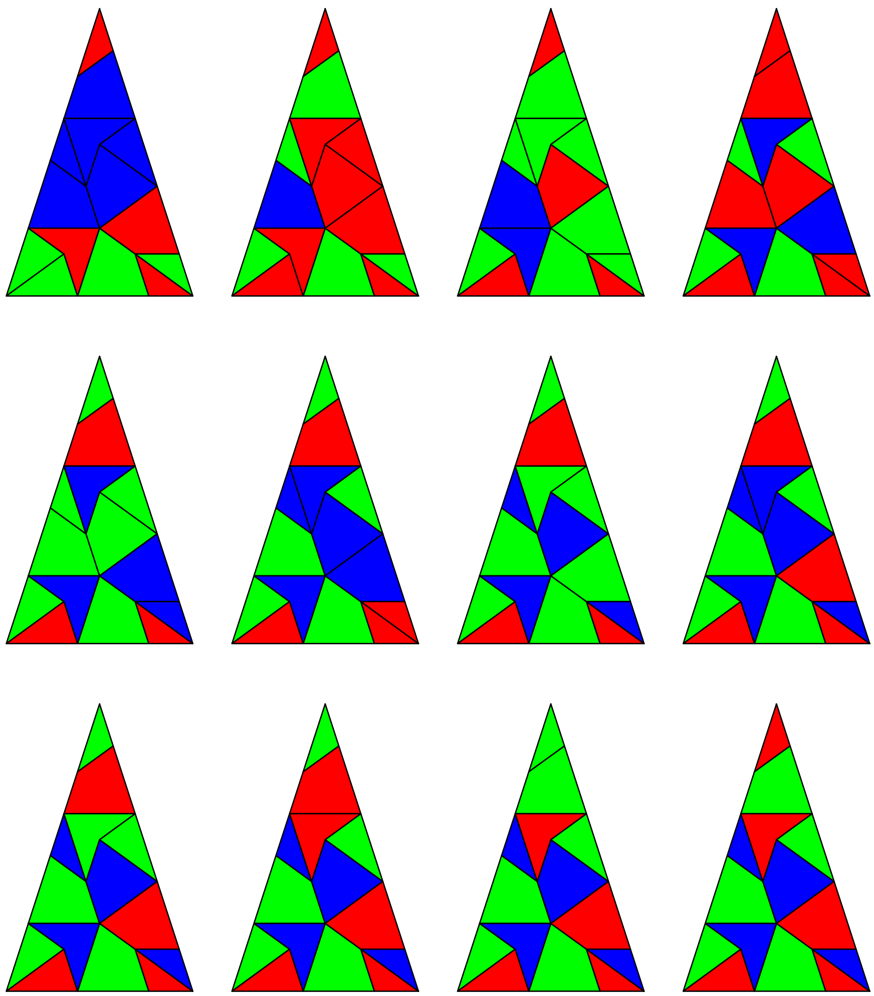

Figure 10: Evolution to a three-coloring